Inspections on the Mandelbrot Sets of Monic One Dimensional Polynomials

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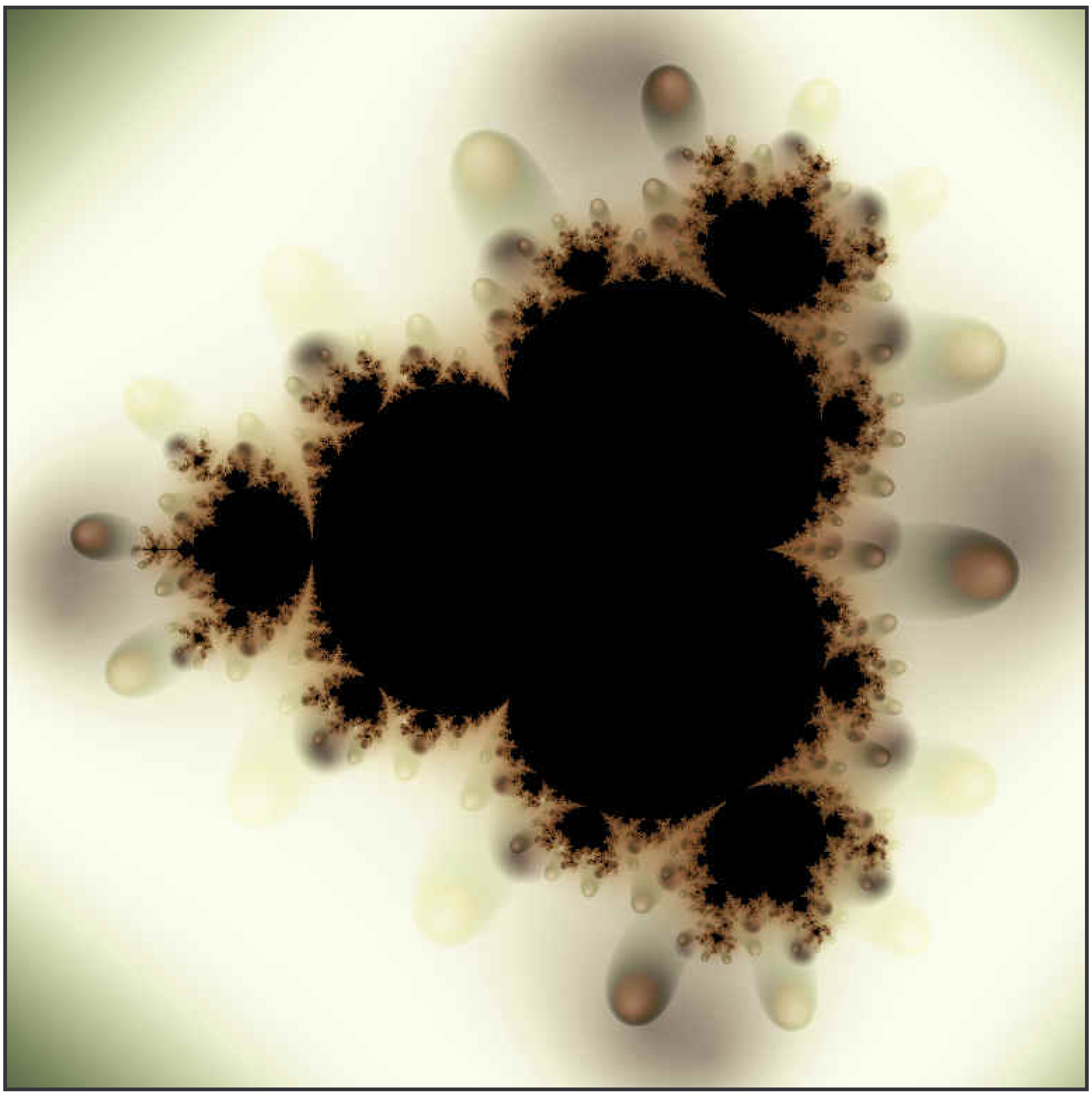
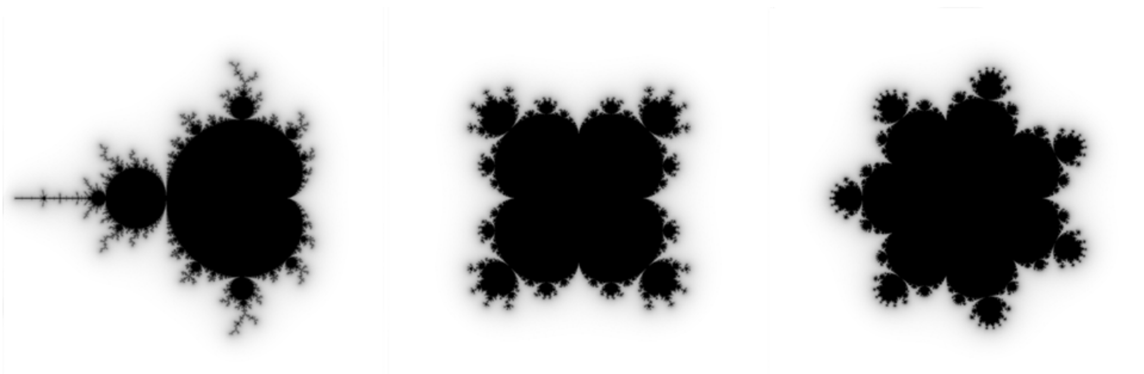


1. Introduction

We are going to concentrate on the Mandelbrot sets for monic one dimensional polynomials of the form $f_{c,k}(z) = z^k + c$. Note that the critical point $f_{c,k}^{'}(z_{crit}) = 0$ is the same for all of them (zero), and that infinity is an super-attractive fixed point. The Mandelbrot set is defined as for the standard case *k*=2.

$$M_k = \left\{ c \in C : \lim_{n \to \infty} f_{c.k}^{\circ n}(0) \neq \infty \right\}$$



M_k sets for k=2, 5 and 8, drawn with a distance estimation algorithm

The set of point created by the iteration of $z_{n+1} = f_{c,k}(z_n)$ with $z_0 = 0$ is called the *orbit* of $z_0 = 0$. Each point in the orbit is thus

$$z_n = \overbrace{f_{c,k}\left(f_{c,k}\left(f_{c,k}(\ldots f_{c,k}(z_o))\right)\right)}^{n \quad times} = f_{c,k}^{\circ n}(z_o)$$

and when applied to $z_0 = 0$ expands the next polynomials in *c*: $z_n = f_{c,k}^{\circ n}(0) = T_k^n(c)$. So, $T_k^0(c) = 0, \quad T_k^1(c) = c, \quad T_k^2(c) = c^k + c$ and so on. Then, a point *c* belongs to M_k if $T_k^n(c)$ remains bounded as $n \to \infty$

$$M_k = \left\{ c \in C : \lim_{n \to \infty} \left| T_k^n(c) \right| \neq \infty \right\}$$

2. Symmetries

After looking to some of the pictures one is tempted to make at least two observations. First, that the set seems to be always symmetric around the real axis. Second, the sets are symmetric under $2\pi \cdot m/(k-1)$ angle rotation. We will quickly show that both are indeed true.

2.1. Vertical Symmetry

Vertical symmetry translates to saying that if a given point c belongs to the M set, then it's conjugate does also: $c \in M_k \Rightarrow c^* \in M_k$. The result comes from the fact that the iteration of $f_{c,k}$ develops a polynomial in c with only real coefficients. But, let's show it anyway by induction.

Let's assume that for one of the iterations we have

$$T_k^n(c) = \left(T_k^n(c^*)\right)^*$$

meaning that if c belongs to M_k, then c^* will also do, since the modulus of a complex number and its conjugate is the same. So, this $T_k^n(c)$ is symmetrical. Let's examine now what happens to the next iteration both for c

$$T_k^{n+1}(c) = \left(T_k^n(c)\right)^k + c$$

and for c^*

$$T_k^{n+1}(c^*) = \left(T_k^n(c^*)\right)^k + c^*$$

From the property that for any complex number w and integer a we have $\left(w^*\right)^a = \left(w^a\right)^*$, we conclude

$$T_k^{n+1}(c^*) = \left(T_k^n(c^*)\right)^k + c^* = \left(\left(T_k^n(c)\right)^k\right)^* + c^* = \left(\left(T_k^n(c)\right)^k + c\right)^* = \left(T_k^{n+1}(c)\right)^*$$

and thus the next iterate is symmetrical too. In other words

$$T_k^n(c) \text{ is symmetrical } \Rightarrow T_k^{n+1}(c) \text{ is symmetrical}$$

We just need to check that $T_k^1(c) = c$ is symmetric to arrive to the conclusion that all the iterates T_k are symmetrical.

$$\left|T_k^n(c)\right| = \left|T_k^n(c^*)\right|, \forall c$$

Therefore the complete M_k set is symmetrical around the real axis:

$$c \in M_k \Rightarrow c^* \in M_k$$

2.2. Rotational Symmetry

For the rotational symmetry we proceed in a similar way. Let's call a $n-symmetry$ to a rotational symmetry of $2\pi \cdot m/n$ radians, for any integers *m* and *n*. Now, we first assume that one iterate $T_k^n(c)$ is (*k*-1)-symmetric as the pictures suggest. That translates to

$$T_k^n(c) \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} = T_k^n(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}})$$

and now we try to demonstrate that next iterate will also be symmetrical. For that we analyze the iteration:

$$T_k^{n+1}(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}) = \left(T_k^n(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}})\right)^k + c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}$$

By the assumption we made,

$$T_k^{n+1}(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}) = \left(T_k^n(c) \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}\right)^k + c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} = \left(T_k^n(c)\right)^k \cdot e^{i\cdot 2\pi \cdot m\cdot k/_{k-1}} + c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}$$

$$= \left(T_k^n(c)\right)^k \cdot e^{i\cdot 2\pi \cdot k/_{k-1}} + c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} = \left(\left(T_k^n(c)\right)^k + c\right) \cdot e^{i\cdot 2\pi \cdot k/_{k-1}} = T_k^{n+1}(c) \cdot e^{i\cdot 2\pi \cdot k/_{k-1}}$$

So, again,

$$T_k^n(c) \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} = T_k^n(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}}) \quad \Rightarrow \quad T_k^{n+1}(c) \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} = T_k^{n+1}(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}})$$

Because we can easily check that for *n*=1 the assumption holds, and because

$$\left|w \cdot e^{i\alpha}\right| = |w|$$

we can say that

$$\left|T_k^n(c)\right| = \left|T_k^n(c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}})\right| \quad \forall n$$

We put it in other way,

$$c \in M_k \Rightarrow c \cdot e^{i\cdot 2\pi \cdot m/_{k-1}} \in M_k$$

And this confirms the rotational symmetry of the pictures. This symmetry, plus the symmetry around the real axis can be exploited to speed up rendering or pixel-counting-based area calculation methods.

3. ***Main bulb parametrization***

As for the classical Mandelbort set *k*=2, the boundary of the main hyperbolic component can be analytically found. The main hyperbolic component or period one hyperbolic component H_k^1 is the subset M_k for which the orbit converges to a fixed point. This means, that for a sufficient large enough *n*,

$$z_{n+1} = f(z_n) = z_n^k + c$$

or by removing the subscript for clarity

$$c = z - z^k$$

Because we want the orbit to converge in practice (ie, that the fixed point is attractive), we need to pay attention to the derivative of the iterated function (respect to the iterated variable) :

$$f'(z) = kz^{k-1}$$

Actually, the fixed point theorem says that we need it's modulus to be less or equal to one

$$|f'(z)| \le 1$$

Intuition suggests that the derivative should be less than one in the inside, and equal to one in the boundary. In fact this was demonstrated by Douady and Hubbard, that made a more complete description on the hyperbolic components (showing for example that all are connected and isomorphic to a disk).

To track the boundary of the main hyperbolic component (also known as main cardioid in the M_2 set), we fix the derivative to have unit length

$$f'(z) = e^{i\omega}$$

so that

$$z = \sqrt[k-1]{\frac{1}{k} e^{i\omega}}$$

Replacing in $c = z - z^k$ we get

$$c_k^1(\omega) = \frac{1}{k^{1/k-1}} e^{i\omega\frac{1}{k-1}} - \frac{1}{k^{k/k-1}} e^{i\omega\frac{k}{k-1}}$$

The subscript *"1"* means that this is the curve in the parameter space (*c*) for the hyperbolic component of period 1.

From this result we see that the curve *c(w)* is not closed at least if the derivative travels in a circle between 0 and 2π. We know the curve must bound a connected set, so we conclude that the derivative must wrap *k-1* times while the parameter loops only once into the complete boundary:

$$c_k^1(\theta) = \frac{1}{k^{1/k-1}} e^{i\theta} - \frac{1}{k^{k/k-1}} e^{ik\theta} \quad 0 \le \theta < \pi$$

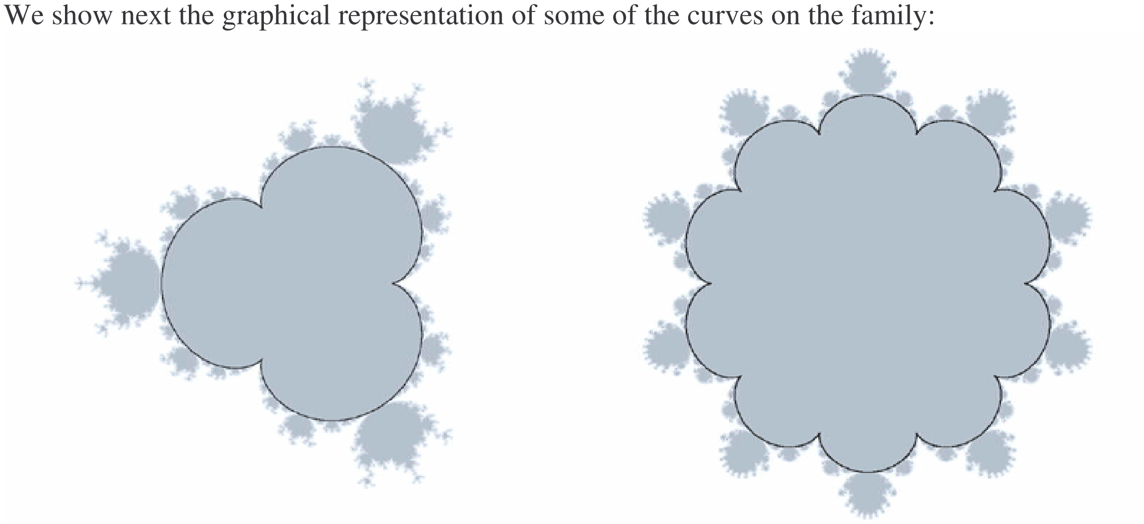
For *k*=2 we get the well known result for the classical Mandelbrot set:

$$c_2^1(\theta) = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{i2\theta}$$

For *k*=3,

$$c_3^1(\theta) = \frac{1}{\sqrt{3}} \left(e^{i\theta} - \frac{1}{3} e^{i3\theta} \right)$$

We show next the graphical representation of some of the curves on the family:



Graph of $c_1^3(\theta)$ *and* $c_1^{11}(\theta)$ *on top of the* M_3 *and* M_{11} *sets*

It's important to note that

$$\lim_{k\to\infty} \frac{1}{k^{1/k-1}} = 1 \quad \text{and} \quad \lim_{k\to\infty} \frac{1}{k^{k/k-1}} = 0$$

so

$$\lim_{k\to\infty} c_k^1(\theta) = e^{i\theta}$$

What means that the curves converge into the unit circle. This will be important later on.

4. ***Analything the curves: internal bound***

For speeding up rendering of the classical Mandelbrot set, a common optimization is to directly detect as internal points those belonging to the main cardioid H_2^1 and the period 2 bulb H_2^2. As shown in [1], these can be easily done checking if either

$$S_2^1(c) = 256 \cdot |c|^4 - 96 \cdot |c|^2 + 32 \cdot \Re\{c\} - 3$$

or

$$S_2^2(c) = 4 \cdot |c+1|^2 - 1$$

are negative. Some formulas has been also derived for the boundary of the period 3 hyperbolic components [2], but they involve complex hyperbolic trigonometric functions, are probably the area involved does not worth the cost of evaluating the function.

For the case of a polynomial of degree 3 or more, things become even worse, so another solution should be found if possible for speeding up rendering as much as possible.

From the description of the curves $c_k^1(\theta)$ we can at least try to calculate the bigger circle that can inscribed in it. It's simple to check that the center of symmetry of the curve is 0 except for *k*=2. Hence we can use the minimum modulus of $c_k^1(\theta)$ as upper bound for the optimization disc.

$$\left|c_k^1(\theta)\right|^2 = \left(\frac{1}{k^{1/(k-1)}}\cos(i\theta) - \frac{1}{k^{k/(k-1)}}\cos(ik\theta)\right)^2 + \left(\frac{1}{k^{1/(k-1)}}\sin(i\theta) - \frac{1}{k^{k/(k-1)}}\sin(ik\theta)\right)^2 =$$

$$= \frac{1}{k^{2/(k-1)}} + \frac{1}{k^{2k/(k-1)}} - \frac{2}{k^{(k+1)/(k-1)}}\left(\cos(i\theta)\cos(ik\theta) + \sin(i\theta)\sin(ik\theta)\right)$$

The function $\cos(i\theta)\cos(ik\theta) + \sin(i\theta)\sin(ik\theta)$ is bounded to ± 1 (what can be checked by showing that it's equivalent to $\cos(t(k-1))$ for example), so we can safely bound the complete expression to

$$\left|c_k^1(\theta)\right|^2 \geq \frac{1}{k^{2/(k-1)}} + \frac{1}{k^{2k/(k-1)}} - \frac{2}{k^{(k+1)/(k-1)}} = (R_k)^2$$

From here we get that

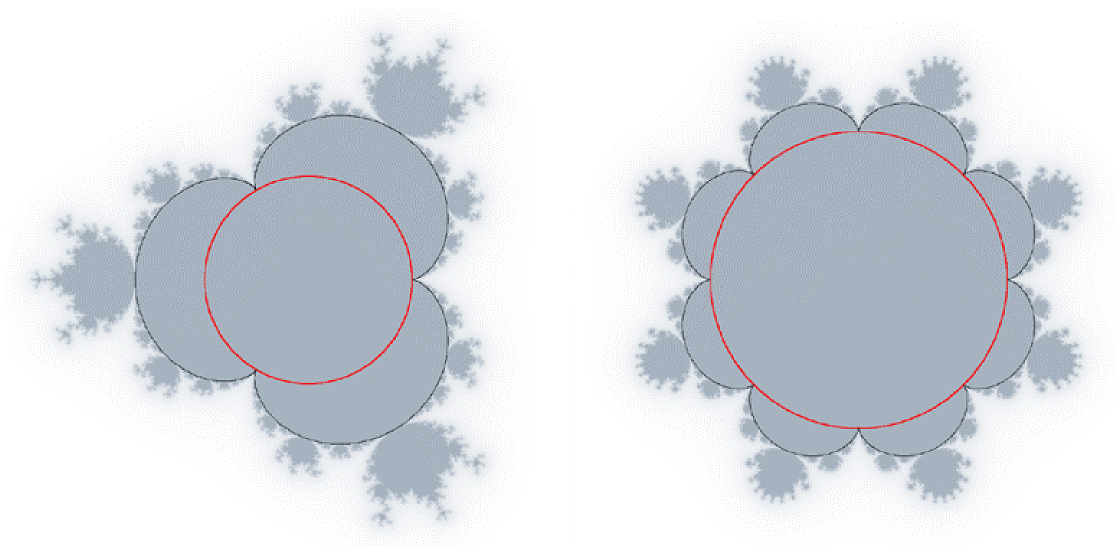
$$R_k = \frac{1}{k^{1/(k-1)}} - \frac{1}{k^{k/(k-1)}} \quad \text{or equivalently} \quad R_k = k^{1/(1-k)} - k^{k/(1-k)}$$

We will call this minimum radius of safety R_k.

Note how

$$\lim_{k\to\infty} R_k = 1$$

We can now draw this safety circles together with the $c_k^1(\theta)$ curves. Notice that the bigger k the better the disk does on saving rendering or pixel counting time.



Circles with radius R_4 *and* R_9 *in red, together with* $c_4^1(\theta)$ *and* $c_9^1(\theta)$ *in black*

5. External Bound

The oldest algorithm ever used for rendering the classic Mandelbrot set is the escape method. Based on the fact that any orbit with a point outside the disk of radius $\max\{|c|,2\}$ escapes to infinity with no "back" stepping anymore, points can be detected to belong to the set as long as the orbit never exceeds this distance from the origin.

The value 2 is not arbitrary. We will try to find the bailout for *k* other than max{c,2}, of the form $B_k = \max\{|c|, Q_k\}$. Assume that $|z_n|$ is already bigger than $|c|$. Escape to infinity will occur then if

$$\frac{|z_{n+1}|}{|z_n|} = \frac{|z_n^k + c|}{|z_n|} > 1 \Rightarrow c \notin M_k$$

Removing the subscripts for clarity, and by using the anti-triangle inequality

$$|a+b| \geq ||a| - |b||$$

we get

$$\frac{|z^k + c|}{|z|} \geq \frac{||z|^k - |c||}{|z|} = |z|^{k-1} - \frac{|c|}{|z|}$$

Since we assumed $|z| > |c|$ we get

$$\frac{|z_n^k + c|}{|z_n|} \geq |z|^{k-1} - \frac{|c|}{|z|} \geq |z|^{k-1} - 1 \geq 1 \Rightarrow |z|^{k-1} \geq 2 \Rightarrow |z| \geq 2^{\frac{1}{k-1}}$$

So, in general we have

$$Q_k = 2^{1/k-1} \quad B_k = \max\{c, 2^{1/k-1}\}$$

Of course this generalizes to the case *k*=2. Note how the escape radius gets smaller as *k* increases. Actually,

$$\lim_{k \to \infty} Q_k = 1$$

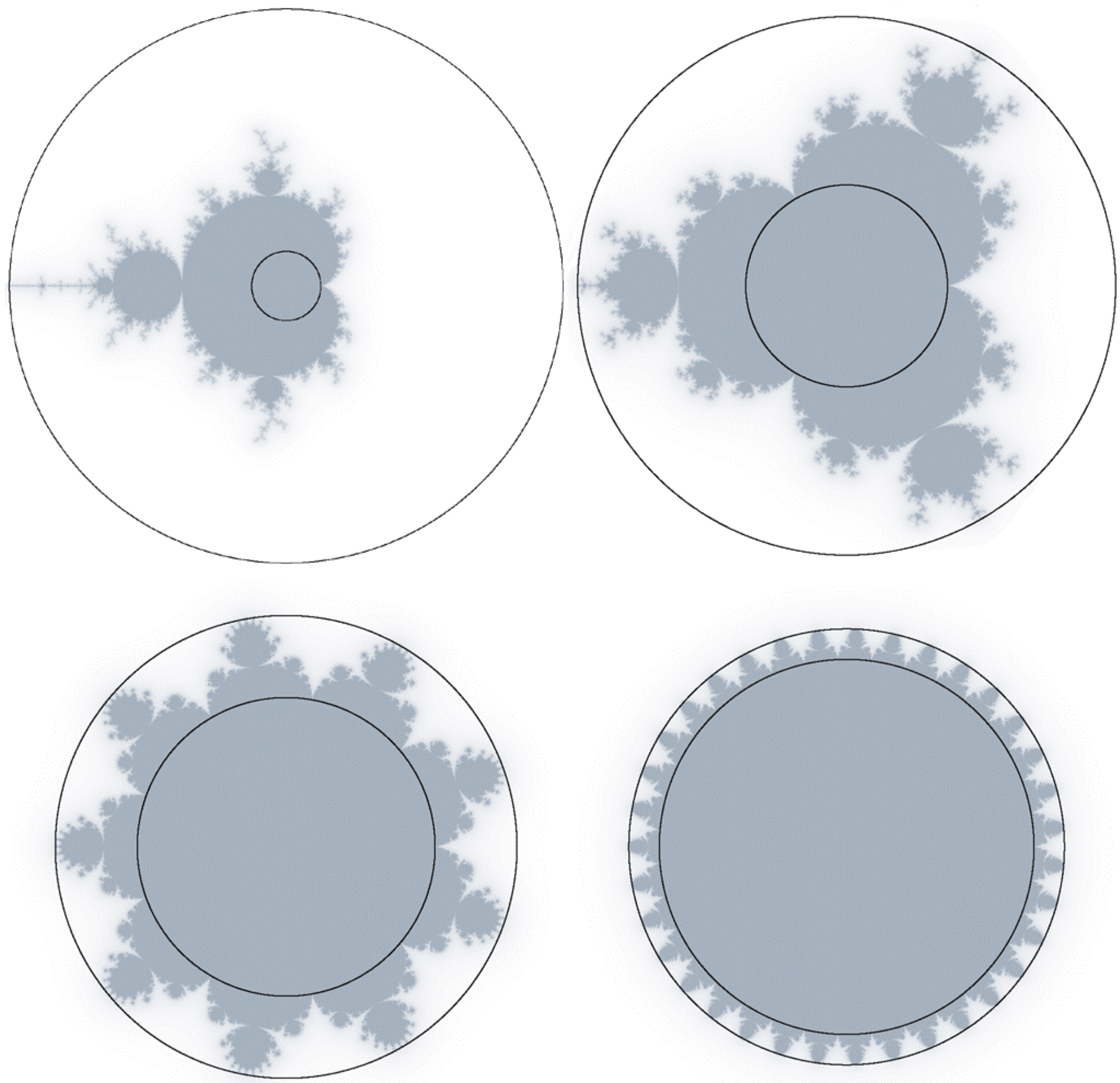
Now that we have the escape radius, we can set an upper bound to the complete set. If $|c| \geq Q_k$ then $B_k = |c|$ is obviously a bounding radius: let's say we start iterating for a point of *c*:

$0, c, c^k + c, ..$

For the second iteration we have

$$\frac{|c^k + c|}{|c|} = |c^{k-1} + 1| \geq ||c^{k-1}| - 1| \geq ||Q_{k-1}| - 1| = Q_{k-1} - 1 \geq 2 - 1 = 1$$

So, the set is completely contained in the disc or radius Q_k .



Circles with radius R_k and Q_k for k=2,4,10 and 36

6. Period 2 bulbs

It's very usual to people to know that for the classic Mandelbrot set the period two hyperbolic component H_2^2 is a disc of radius 1/4 centered at –1. Can we find also such a description for *k* other than 2? In [1] a direct method was used to derive both $c_2^2(\theta)$ and S_2^2. Let's show first how the case *k*=2 can be done in another way that will allow to generalize the method for *k*>2.

6.1. Case for *k*=2

We start by imposing that the after some iteration *n*, the map $f_c^{\circ 2}(z)$ must have a fixed point:

$$z = (z^2 + c)^2 + c \quad \rightarrow \quad z^4 + 2c \cdot z^2 - z + (c^2 + c) = 0$$

We know that points in the main hyperbolic component have an converging orbit, and since convergence can be considered as being periodic of period 2 as well, we have to extract the hidden solution that represents H_2^1 from the previous polynomial.

For period 1 we have

$$z = f_c(z) \quad \rightarrow \quad z^2 - z + c = 0$$

So we do the division

$$\frac{z^4 + 2c \cdot z^2 - z + (c^2 + c)}{z^2 - z + c} = z^2 + z + (c+1) = 0$$

The derivative must be analyzed as for ensuring the convergence:

$$\lambda = \left(f_c^{\circ 2}\right)'(z)$$

and by the chain rule

$$\lambda = f_c'(z) \cdot f_c'(f_c(z)) = 2z \cdot 2(z^2 + c)$$

For easier later manipulation, we make the change of variable

$$\gamma = \lambda / 4$$

The boundary of H_2^2 in the *c* plane - $c_2^2(\theta)$ - will be given by the fixed point theorem's constraint $|\lambda| = e^{i\theta}$ (and the interior when the modulus is less than one and hence the iteration converges) so we should be able to isolate *z* from one of the equations, and substitute in the other one to get *c* as a function of λ. There is a technique called "resultant elimination" that is quite simple to use:

$$\begin{cases} z^2+z+(c+1)=0 \\ z^3+cz-\gamma=0 \end{cases} \rightarrow \begin{vmatrix} 1 & 1 & 1+c & 0 & 0 \\ 0 & 1 & 1 & 1+c & 0 \\ 0 & 0 & 1 & 1 & 1+c \\ 1 & 0 & c & -\gamma & 0 \\ 0 & 1 & 0 & c & -\gamma \end{vmatrix} = (c+1-\lambda)^2=0$$

We now undo the change of variable

$$c=-1+\lambda/4$$

and impose the boundary condition to get

$$c_2^2(\theta)=-1+\frac{e^{i\theta}}{4}$$

6.2 Case for *k*=3

Let's follow exactly the same steps for the case *k*=3. The second iterate must be equal to the iterated value, so

$$z=f_c^{\circ 2}(z) \quad \rightarrow \quad z^9+3cz^6+3c^2z^3-z+(c^3+c)=0$$

Again, this should be divisible by the period 1 polynomial

$$z=f_c(z) \quad \rightarrow \quad z^3-z+c=0$$

$$\frac{z^9+3cz^6+3c^2z^3-z+(c^3+c)}{z^3-z+c}=z^6+z^4+2cz^3+z^2+cz+(c^2+1)=0$$

For the derivative,

$$\lambda=\left(f_c^{\circ 2}\right)'(z)=f_c'(z)\cdot f_c'(z^3+c)=9z^2\cdot(z^3+c)^2$$

We make the change of variable

$$\gamma^2=\lambda/9 \quad \rightarrow \quad z\cdot(z^3+c)-\gamma=0$$

Applying the resultant elimination method results in a 11x11 determinant this time:

$$\begin{cases} z^6 + z^4 + 2cz^3 + z^2 + cz + (c^2+1) = 0 \\ z^4 + cz - \gamma = 0 \end{cases} \rightarrow \quad |\;| \quad = \quad \left(c^2 - (\gamma-1)(\gamma+1)^2\right)^2 = 0$$

We undo the change of variable and we get

$$c = \sqrt{-1 + \frac{1}{27}\lambda(\lambda^2 + 3\lambda - 9)}$$

hence

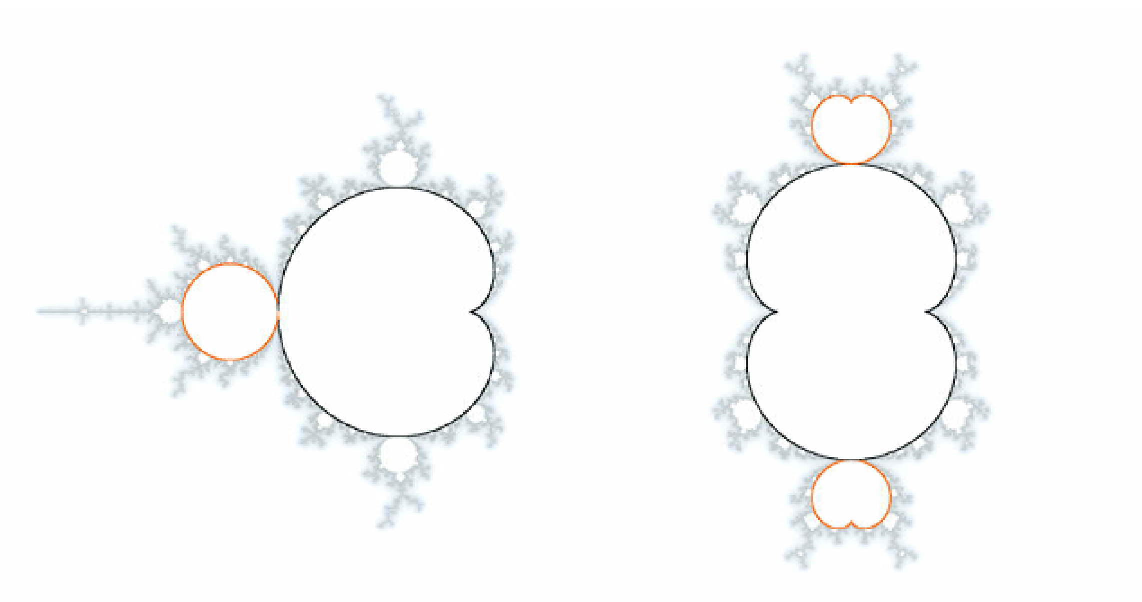
$$c_3^2(\theta) = \sqrt{-1 + \frac{1}{27}e^{i\theta}(e^{i2\theta} + 3e^{i\theta} - 9)}$$

Because of the square root we can conclude that there will be two hyperbolic components of period 2, both symmetrical around the origin.

Comparing the formulas for the $c_{2,3}^2(\theta)$ one is tempted to believe that in general there will be *k-1* hyperbolic components of period two, with a formula of the type

$$c_k^2(\theta) = \sqrt[k-1]{-1 + \frac{1}{k^k}e^{i\theta}(...)}$$

Anyway, we can at least draw the curves for *k*=2 and *k*=3:



$c_2^2(\theta)$ and $c_3^2(\theta)$ in orange together with $c_2^1(\theta)$ and $c_3^1(\theta)$ in black

7. *Limit of the Area of the Mandelbrot set*

We have already shown that H_k^1 component tends to the unit circle as k increases to infinity, while the outer bounding disk has becomes smaller and smaller with a radius 1 as limit. In general we have that, if A_k is the area of the Mandelbrot set M_k.

$$\pi \cdot {R_k}^2 \leq A_k \leq \pi \cdot {Q_k}^2 = \pi \cdot 2^{2/k-1}$$

We showed that

$$\lim_{k\to\infty} R_k = 1^-$$

$$\lim_{k\to\infty} Q_k = 1+$$

so we conclude that

$$\lim_{k\to\infty} A_k = \pi$$

8. ***Area of the main Hyperbolic component***

Working with hyperbolic components of period bigger than two is not easy. Even period two becomes problematic for *k*>3. However, the main one is easy to deal with.

Recall that we got the following result for boundary of H_k^1:

$$c_k^1(\theta) = \frac{1}{k^{1/k-1}} e^{i\theta} - \frac{1}{k^{k/k-1}} e^{ik\theta}$$

We can calculate the area inside of the curve by using the Green's theorem from potential theory, that is basically the same as the Stokes theorem applied to 2D.

$$\int_H \left(\partial Q/\partial x - \partial P/\partial y \right) \cdot dx \cdot dy = \int_{\partial H} (Pdx + Qdy)$$

This basically means that we can replace an integral over a surface by an integral over the boundary of that surface. It's obviously the perfect case for what we need, because the area is basically the integral of the constant function 1 over the surface, and we indeed know the boundary curve for it. It's a classical result that by making P=-y and Q=x, the theorem transforms into

$$2\int_H dx \cdot dy = 2A = \int_{\partial H} (xdy - ydx) \quad \Rightarrow \quad A = \frac{1}{2} \int_{\partial H} (xdy - ydx)$$

Since our curve is parametric, we can do

$$c_1^k(\theta) = x(\theta) + i \cdot y(\theta)$$
$$dx = x'(\theta) \cdot d\theta$$
$$dy = y'(\theta) \cdot d\theta$$

$$A = \frac{1}{2} \int_0^{2\pi} (x \cdot y' - y \cdot x') d\theta$$

with

$$x(\theta) = \frac{1}{k^{1/k-1}} \cos(\theta) - \frac{1}{k^{k/k-1}} \cos(k\theta) \quad \rightarrow \quad x'(\theta) = \frac{1}{k^{1/k-1}} \left(\sin(k\theta) - \sin(\theta) \right)$$
$$y(\theta) = \frac{1}{k^{1/k-1}} \sin(\theta) - \frac{1}{k^{k/k-1}} \sin(k\theta) \quad \rightarrow \quad y'(\theta) = \frac{1}{k^{1/k-1}} \left(-\cos(k\theta) + \cos(\theta) \right)$$

$$x(\theta) \cdot y'(\theta) - y(\theta) \cdot x'(\theta) = 2(k+1) k^{\frac{k+1}{k-1}} \sin^2\left(\frac{1}{2} \theta (k-1) \right)$$

so

$$A_k^1 = \frac{1}{2}\int_0^{2\pi}(x \cdot y' - y \cdot x')d\theta = \pi \frac{(k+1)}{k^{\frac{k+1}{k-1}}}$$

that we rewrite as

$$A_k^1 = \pi(k+1)k^{-\frac{k+1}{k-1}}$$

For the standard Mandelbrot set, the area of the main cardioid is then $A_2^1 = 3\pi/8$

There are the values for the first *k*:

$$A_k^1 = \left\{\frac{3}{8}\pi, \frac{4}{9}\pi, \frac{5}{4^{5/3}}\pi, \ldots\right\}$$

It's clear also that

$$\lim_{k\to\infty} A_k^1 = \pi$$

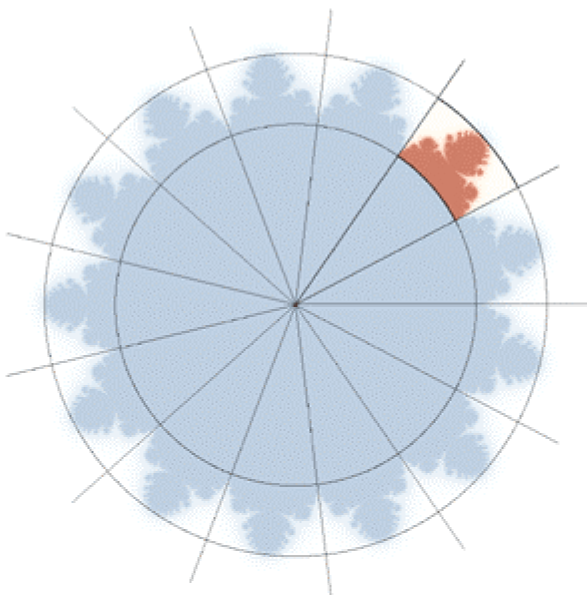
Recall that we showed the complete area of the set is also one in the limit. As consequence we can conclude that the relative area of the rest of hyperbolic components vanish as the degree of the polynomial increases. This is something that agrees with the graphical experiments.

$$\lim_{k\to\infty} A_k^h = \begin{cases} 1 & if \quad h=1 \\ 0 & if \quad h>1 \end{cases}$$

9. ***Experiments with pixel counting***

Much has been discussed about the area of the standard Mandelbrot set. Only one mathematically exact solution has been proposed, but it happens to be far too slow to use in any computer. The area is given as a summation series, may be a bit disappointing for many people that would expect a nice small and compact formula, probably involving a few universal constants (π, Feigenbaun number, e, …). There has been many experiments in other directions, thou, as pixel counting approaches (sometimes assisted by the distance estimator to better bound the error) [3].

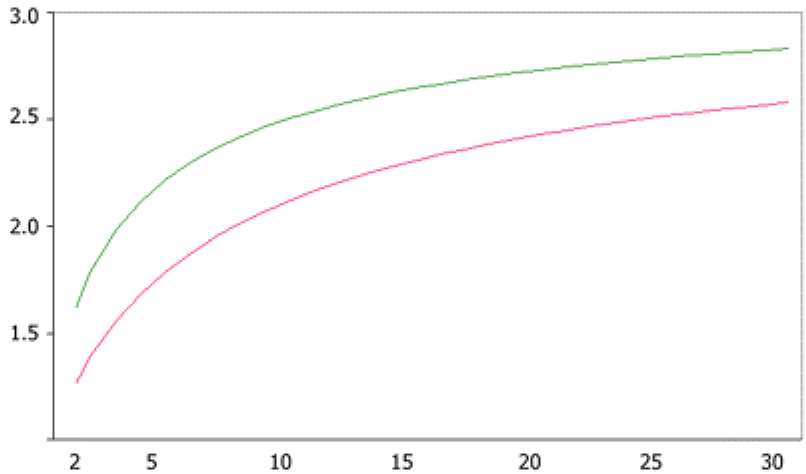
Here we will not care that much about the exact value of the area for the standard Mandelbrot set, but the behavior of the area as *k* increases instead. We know the area is asymptotically approaching π. We also know, as discussed before, that the period one hyperbolic component influences more and more in the total area, so the graph should asymptotically behave as the formula we got for A_k^1.



Only in the reddish region the pixel counting is done

This is the results after a low resolution (4096^2 px, $5 \cdot 10^4$ it.) pixel counting method. However, the knowledge of R_k and Q_k was used, as well as the rotational symmetry of the sets to speed up the calculation time. The table below shows some results from *k*=2 to *k*=31. No more than two decimals are probably correct.

1.5065	1.7959	1.9828	2.1167	2.2183	2.2986	2.3642	2.4188
2.4655	2.5052	2.5405	2.5712	2.5984	2.6230	2.6451	2.6650
2.6835	2.7000	2.7157	2.7299	2.7432	2.7553	2.7669	2.7778
2.7876	2.7970	2.8059	2.8143	2.8223	2.8299		



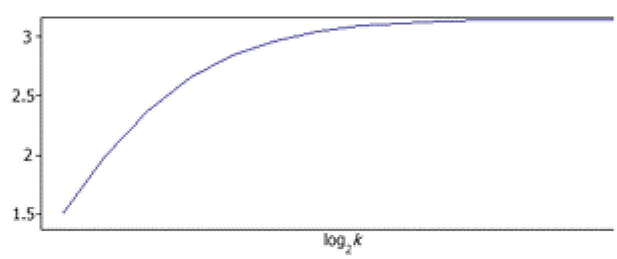
Approximated values of A_k in green, and exact values for A_k^1 in red

The green graph comes from the experimental results. The red one is the represents the area of the period one hyperbolic component H_k^1 calculated with the formula we got for A_k^1. As expected, the red graph gets closer to the green one as k increases.

To better understand the behavior for big k, we can also calculate the area for the first powers of two. For k>16 a grid of 16384^2 was used with 10^5 iterations. As expected, areas approach π.

1.5065	1.9829	2.3642	2.6451	2.8369	2.9606	3.0366
3.0816	3.1081	3.1231	3.1313	3.1358	3.1379	3.1387

Area values for $A_2, A_4, A_8, A_{16}, \ldots A_{16384}$



A_k graph, with logarithmic scale

10. References

[1] "De la geometría a la dinamica", www.rgba.org/iq/fractals/research/arquimedes.pdf
[2] "A Parametrization of the Period 3 Hyperbolic Component of the Mandelbrot Set", Dante Giarrusso, Yuval Fisher, 1994
[3] "Bounding the Area of the Mandelbrot Set", Yuval Fisher, Jay Hill